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1999 J. Phys. A: Math. Gen. 32 5611

(http://iopscience.iop.org/0305-4470/32/30/308)

J. Phys. A: Math. Gen. **32** (1999) 5611–5625. Printed in the UK PII: S0305-4470(99)03646-X

Motions in a bose condensate: VI. Vortices in a nonlocal model

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Received 22 April 1999

Abstract. Nonlocal nonlinear Schrödinger equations are considered as models of liquid helium II. The models contain a nonlocal interaction potential that leads to a phonon–roton-like dispersion relation. Also, a higher-order term in the local density approximation for the correlation energy is introduced into the model to overcome nonphysical mass concentrations. These equations are solved for straight-line vortices. It is demonstrated that the parameters of the equation can be chosen to bring into agreement the vortex core parameter and the healing length. The structure of vortex rings of large radius is studied. The family of the vortex rings of different radii propagating with different velocities is found numerically. As the velocity of the vortex ring reaches the Landau critical velocity the sequence of rings terminates.

1. Introduction

This is the sixth in a series of papers devoted to the Bose condensate as applied to superfluid helium and especially superfluid vortices; see Roberts and Grant (1971), Grant (1973), Grant and Roberts (1974), Jones and Roberts (1982), and Jones *et al* (1986). These will be referred to below, as papers I–V, respectively.

As liquid helium is cooled through 2.17 K it undergoes a phase transition to a superfluid state and remains in this state down to 0 K at the vapour pressure. Helium at 0 K has large interatomic spacing and is often described in terms of a weakly interacting Bose gas. The imperfect Bose condensate in the Hartree approximation is governed by equations that were derived by Gross and by Ginsburg and Pitaevskii. In terms of the single-particle wavefunction $\psi(\boldsymbol{x}, t)$ for N bosons of mass M, the time-dependent self-consistent field equation is

$$\mathrm{i}\hbar\psi_t = -\frac{\hbar^2}{2M}\nabla^2\psi + \psi\int|\psi(\boldsymbol{x}', t)|^2 V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' \tag{1}$$

where $V(|\boldsymbol{x}-\boldsymbol{x}'|)$ is the potential of the two-body interactions between bosons. The normalization condition is

$$\int_V |\psi|^2\,\mathrm{d}\boldsymbol{x} = N. \tag{2}$$

The Madelung transformation for the mass probability density ρ

$$\rho = M\psi\psi^* \tag{3}$$

and for the mass flux $\boldsymbol{j} = \rho\boldsymbol{v}$

$$\boldsymbol{j} = \frac{\hbar}{2\mathrm{i}}(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*) \tag{4}$$

0305-4470/99/305611+15$30.00

converts (1) into equations of continuity and motion. The internal energy per unit volume, $\mathcal{E}$, at point $\boldsymbol{x}$ and time t is given by

$$\mathcal{E}(\rho) = \frac{\hbar^2}{8M^2\rho}(\nabla\rho)^2 + \frac{1}{2M^2}\int \rho(\boldsymbol{x})V(|\boldsymbol{x}-\boldsymbol{x}'|)\rho(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' \tag{5}$$

and the total energy, W, is

$$W = \int \mathcal{E}(\rho)\,\mathrm{d}\boldsymbol{x} = \int \frac{\hbar^2}{8M^2\rho}(\nabla\rho)^2\,\mathrm{d}\boldsymbol{x} + W_c(\rho). \tag{6}$$

The first term on the right-hand side of (6) describes the quantum kinetic energy of a Bose gas of nonuniform density; $W_c(\rho)$ is a potential or correlation energy that incorporates the effect of interactions. For a weakly interacting Bose system (1) is simplified by replacing $V(|\boldsymbol{x}-\boldsymbol{x}'|)$ with a δ-function repulsive potential of strength $W_0 = \int U\,\mathrm{d}\boldsymbol{x}'$, since we can consider that the wavefunction, $\psi(\boldsymbol{x}, t)$, changes very slowly on atomic distances:

$$\mathrm{i}\hbar\psi_t = -\frac{\hbar^2}{2M}\nabla^2\psi + W_0\psi|\psi|^2. \tag{7}$$

Some aspects of superfluid behaviour, such as the annihilation of vortex rings (see paper IV), the nucleation of vortices (Frisch *et al* 1992), and vortex line reconnection (Koplik and Levine 1993, 1996) are captured by this local model. At the same time the dispersion relation between the frequency, ω, and wavenumber, k, of sound waves according to (7) is

$$\omega^2 = c^2k^2 + \left(\frac{\hbar}{2M}\right)^2 k^4 \tag{8}$$

where

$$c = (W_0\rho_\infty)^{\frac{1}{2}}/M. \tag{9}$$

This dispersion relation has no roton minimum, which is held responsible for many of the properties of a superfluid. The natural way of incorporating the correct phonon–roton-like spectrum would be to consider a more general model (1) with a realistic two-particle potential, V, that leads to a dispersion relation close to experimental reality. Unfortunately, as was shown by Berloff (1999), under the very minimum requirements on such a potential, such as the correct position of the roton minimum and the correct bulk normalization, the general model (1) has nonphysical features, such as the loss of hyperbolicity leading to the creation of nondissipative mass concentrations.

A more accurate approach in modelling liquid helium is through density-functional theory (Dalfovo *et al* 1995), which attempts to give an adequate microscopic description of interactions. In this approach the total energy (6) is still written as a functional of the one-body density, but it includes short-range correlations (Dupont-Roc *et al* 1990). This approach has provided a quantitatively and qualitatively reliable representation of the superfluid properties of free surfaces, helium films, and droplets (see Dalfovo *et al* 1995 and references therein). At the same time this approach is phenomenological and results in rather complicated forms of the energy functionals with many parameters that are chosen to reproduce liquid helium properties.

Our goal is to modify the nonlocal model (1) in the spirit of a density-functional approach, but to restrict ourselves to only one additional nonlinear term in the expression for the correlation energy. This allows us to remedy the nonphysical features of model (1), while not only retaining an adequate representation of the Landau dispersion relation, but also simplicity in the analytical and numerical studies. One of the main objectives is to elucidate the properties of vortex rings.

2. Nonlocal model

The correlation energy of the Skyrme interactions in nuclei (Vautherin 1972) is given by

$$W_c(\rho) = \frac{1}{M^2}\int\left[\frac{W_0}{2}\rho^2 + \frac{W_1}{2+\gamma}\rho^{2+\gamma} + W_2(\nabla\rho)^2\right] \tag{10}$$

where W_0, W_1, W_2 and γ are phenomenological constants. The first two terms give a local density approximation, and the gradient term corresponds to finite-range interactions. In a somewhat similar way to Dupont-Roc *et al* (1990), we add the necessary nonlocality of interactions directly into the first term of (10) by introducing a two-body interaction potential, $V(|\boldsymbol{x}-\boldsymbol{x}'|)$, so that (10) becomes

$$W_c(\rho) = \frac{1}{M^2}\int\left[\frac{1}{2}\int \rho(\boldsymbol{x})V(|\boldsymbol{x}-\boldsymbol{x}'|)\rho(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' + \frac{W_1}{2+\gamma}\rho^{2+\gamma}\right]\mathrm{d}\boldsymbol{x}. \tag{11}$$

This incorporates and generalizes the W_2 interaction term in (10), which has therefore been abandoned. The differences in our expression for the correlation energy and the one used by Dupont-Roc *et al* (1990) and Dalfovo *et al* (1995) are, first, that we keep the higher-order term of the local density approximation unchanged and second, that the two-body interaction potential is not assumed to be the standard Lennard-Jones potential; instead, $V(|\boldsymbol{x}-\boldsymbol{x}'|)$ will be chosen so that the implied dispersion relation gives a good fit to the Landau dispersion curve. Following Jones (1993) we consider a potential of the form

$$V(|\boldsymbol{x}-\boldsymbol{x}'|) = V(r) = (\alpha + \beta A^2r^2 + \delta A^4r^4)\exp(-A^2r^2) \tag{12a}$$

and the slightly modified potential

$$V(|\boldsymbol{x}-\boldsymbol{x}'|) = V(r) = (\alpha + \beta A^2r^2 + \delta A^4r^4)\exp(-A^2r^2) + \eta\exp(-B^2r^2) \tag{12b}$$

where A, B, α, β, δ and η are parameters that can be chosen to give excellent agreement with the experimentally determined dispersion curve.

On adopting (11), we find that the nonlinear Schrödinger equation replacing (1) is

$$\mathrm{i}\hbar\psi_t = -\frac{\hbar^2}{2M}\nabla^2\psi + \frac{\psi}{M}\int|\psi(\boldsymbol{x}',t)|^2V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' + \frac{\psi W_1}{M}|\psi|^{2(1+\gamma)}. \tag{13}$$

If E_v is the average energy level per unit mass of a boson, we write

$$\Psi = \exp(\mathrm{i}ME_vt/\hbar)\psi \tag{14}$$

so that (13) becomes

$$\mathrm{i}\hbar\Psi_t = -\frac{\hbar^2}{2M}\nabla^2\Psi + \frac{\Psi}{M}\left(\int|\Psi(\boldsymbol{x}',t)|^2V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' + W_1|\Psi|^{2(1+\gamma)} - ME_v\right). \tag{15}$$

Casting this equation into dimensionless form by the transformation

$$\boldsymbol{x} \to \frac{\hbar}{(2M^2E_v)^{1/2}}\boldsymbol{x} \qquad t \to \frac{\hbar}{2ME_v}t \tag{16}$$

we obtain

$$-2\mathrm{i}\frac{\partial\Psi}{\partial t} = \nabla^2\Psi + \Psi\left[1 - \int|\Psi(\boldsymbol{x}')|^2V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' - \chi|\Psi|^{2(1+\gamma)}\right] \tag{17}$$

where the nondimensional parameter χ is given by $\chi = W_1\rho_\infty^{1+\gamma}/M^2E_v$ and the nondimensional constant in front of the integral was absorbed into V. The bulk normalization condition is

$$\int V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' = 1-\chi \tag{18}$$

or

$$4\pi \int_0^\infty V(r) r^2\, \mathrm{d}r = 1 - \chi. \tag{19}$$

To determine the dispersion relation we linearize the solution of Ψ about the rest state $\Psi = 1$. We write $\Psi = 1 + \epsilon\Psi_r' + \epsilon\Psi_i'$, where Ψ_r' and Ψ_i' are real and imaginary parts of Ψ', respectively, and consider plane waves of the form $\Psi_r' = \exp \mathrm{i}(\omega t - kx)$ for $\epsilon \ll 1$. Then the dispersion relation can be written as

$$\omega^2 = \tfrac{1}{4}k^4 + 2\pi k \int \sin kr\, V(r) r\, \mathrm{d}r + \tfrac{1}{2}(1+\gamma)\chi k^2. \tag{20}$$

The bulk normalization condition (18) gives the slope at the origin (the dimensionless speed of sound) as $\sqrt{(1+\gamma\chi)/2}$. Since the known speed of sound is approximately 238 m s^{-1}, the unit of length (healing length) of our model is $[L] = 0.471\sqrt{1+\gamma\chi}$ Å and the unit of time is $[t] = 1.4 \times 10^{-13}(1+\gamma\chi)$ s. The parameters α, β and δ of the nonlocal potential (12a) are chosen so that the bulk normalization condition (19) is satisfied and the dispersion relation has the position of the roton minimum close to that experimentally observed at the vapour pressure $k_{rot} = 1.926$ Å^{-1}, $\omega_{rot} = 8.62\ \mathrm{K}k_B/\hbar$ (Donnelly *et al* 1981), which in our nondimensional units is at $k_{rot} = 0.9077\sqrt{1+\gamma\chi}$, $\omega_{rot} = 0.158(1+\gamma\chi)$. The free parameter A in (12a) is chosen with three requirements in mind: (i) the entire dispersion curve (20) gives a reasonable fit to the Landau dispersion curve, (ii) two-particle interactions exhibit a strong repulsion at close distances, and (iii) the potential $V(r)$ is nonzero on the smallest possible interval, in order to make the numerics tractable.

The controversy in the literature (see Brooks and Donnelly 1977 and references therein) about the form of the dispersion curve at low momenta has now been settled, and it is generally accepted that the dispersion relation has a positive k^3 term (the dispersion curve at the origin is concave up) until the pressure reaches some threshold at which the second derivative of ω at the origin changes sign. The potential (12a) implies a negative k^3 term and the coefficients η and B in (12b) can be chosen so that the resulting dispersion relation has a positive k^3 term, for instance, the same as the Bogoliubov spectrum (8). At the same time, to obtain the roton minimum, B must be much smaller than A, and this makes the potential more nonlocal and less amenable for numerical work. In (12b) for $\chi = 3.5$ and $\gamma = 1$, we took $A = 1.6$, $B = 1$, $\alpha \approx 1.9123$, $\beta \approx -28.9815$, $\delta \approx 9.5$, and $\eta \approx 1$. We analysed the problem (17) for two possible choices of the parameter γ. First, we can view the term $W_1\rho^{2+\gamma}$ in (10) as the second term in the nonlinear expansion of the correlation energy $E_c(\rho)$ the in powers of ρ, and that yields $\gamma = 1$ (cf the expression for the Hamiltonian in Dalfovo *et al* (1995)). The second possible choice is to take $\gamma = 2.8$, which gives the velocity, c, of long-wavelength sound waves proportional to $\rho^{2.8}$. This brings about agreement with the experimentally determined Grüneisen constant $U_G = (\rho\partial c/\partial\rho c)_T \approx 2.8$ (Brooks and Donnelly 1977 and references therein).

3. Rectilinear vortex

Jones (1993) computed the structure and energy per unit length of the straight-line vortex of the nonlocal model (1) with the potential (12a). The approach of the fluid density (3) to the uniform state at infinity was shown to be oscillatory rather than monotonic, in a similar way to the observations (Sadd *et al* 1997). The energy per unit length is considerably reduced compared with that of the local model (7), and is in better agreement with the results of experiments on vortex rings of large radius (Rayfield and Reif 1964). Nevertheless, this model failed to bring the vortex core parameter (see (34) below) and the healing length into agreement.

In this section we shall conduct an analysis similar to that of Jones but for the straight-line vortex of the nonlocal model (17) for both $\gamma = 1$ and $\gamma = 2.8$.

In cylindrical polar coordinates (r, θ, z) the nonlocal model (17) takes the form:

$$-2\mathrm{i}\Psi_t = \Psi_{rr} + \frac{1}{r}\Psi_r + \frac{1}{r^2}\Psi_{\theta\theta} + \Psi_{zz} + \Psi\biggl(1 - \chi|\Psi|^{2(1+\gamma)} - \int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} |\Psi(r', \theta', z')|^2 \times V\left(\sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta') + (z - z')^2}\right) r'\,\mathrm{d}\theta'\,\mathrm{d}z'\,\mathrm{d}r'\biggr). \tag{21}$$

The equation for the amplitude of the steady straight-line vortex $R(r)$ is found by substituting $\Psi = R(r)\exp\mathrm{i}\theta$ into (21) and integrating the nonlocal potential in θ' and z':

$$R''(r) + \frac{1}{r}R'(r) - \frac{R(r)}{r^2} + R(r) - \chi R(r)^{2(1+\gamma)+1} = \frac{2\pi^{3/2}}{A}R(r)\int_0^\infty R(r')^2 \exp(-A^2(r^2 + r'^2))[g_0 I_0(\sigma) - g_1 I_1(\sigma)]r'\,\mathrm{d}r' \tag{22}$$

where

$$g_0 = \alpha + \frac{\beta}{2} + \frac{3\delta}{4} + A^2(\beta + \delta)(r^2 + r'^2) + \delta A^4((r^2 + r'^2)^2 + 4r^2 r'^2) \tag{23}$$

$$g_1 = \sigma(\beta + 2\delta) + 4\delta A^4 rr'(r^2 + r'^2) \qquad \sigma = 2A^2 rr'. \tag{24}$$

In (22), I_n is the modified Bessel function of order n. Analysis of (22) shows that $R(r) \approx 1 - 1/2r^2$ as $r \to \infty$.

Equation (22) was solved iteratively using a finite-difference method. Figure 1 gives plots of the relative density $\rho/\rho_0 = R^2$ as functions of the distance from the centre of the vortex for the local model (7), the nonlocal model (17) with the potential (12*b*) for $\gamma = 1$, $\chi = 3.5$,

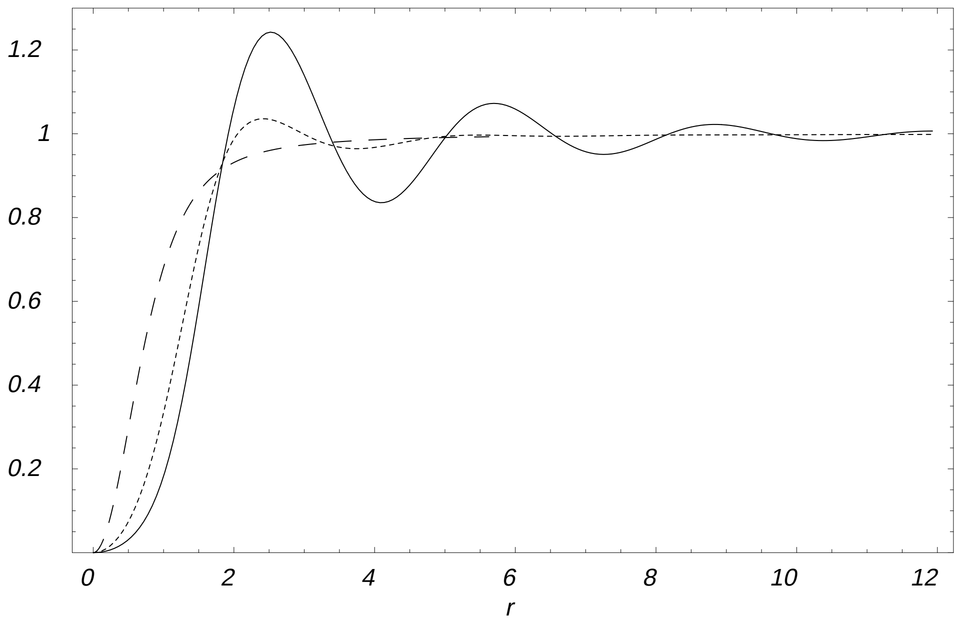

Figure 1. The relative density ρ/ρ_0 as functions of the distance from the centre of the vortex for the local model (7) (dashed curve), the nonlocal model (17) with the potential (12*b*) for $\gamma = 1$, $\chi = 3.5$, $\delta = 1$ (solid curve) and for $\gamma = 2.8$, $\chi = 1.25$, $\delta = -5$ (dotted curve).

$\delta = 1$ and for $\gamma = 2.8$, $\chi = 1.25$, $\delta = -5$. The graph for the nonlocal model (17) with $\gamma = 1$ is in a good agreement with the results of variational Monte Carlo calculations (Sadd *et al* 1997).

To determine the energy of the vortex in the condensate we restore dimensional units temporarily. Following Jones and Roberts (paper IV) we denote by Ψ_u the wavefunction of the undisturbed system of the same mass, so that

$$\int_V |\Psi|^2 \, \mathrm{d}\boldsymbol{x} = \Psi_u^2 v \tag{25}$$

where $v = \int_V \mathrm{d}\boldsymbol{x}$, and by Ψ_∞ the wavefunction of the bulk: $\Psi \to \Psi_\infty$ as $r \to \infty$. The energy of the system is

$$\mathcal{E} = \frac{\hbar^2}{2M} \int_V |\nabla\Psi|^2 \, \mathrm{d}\boldsymbol{x} + \frac{1}{2M} \int_V (\Psi_\infty^2 - |\Psi|^2) V(|\boldsymbol{x} - \boldsymbol{x}'|)(\Psi_\infty^2 - |\Psi(\boldsymbol{x}')|^2) \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{x}$$
$$+ \frac{W_1}{M} \left[\int_V |\Psi|^{2(2+\gamma)} \, \mathrm{d}\boldsymbol{x} - \Psi_u^{2(2+\gamma)} v \right]. \tag{26}$$

The last term on the right-hand side can be written

$$\int_V |\Psi|^{2(2+\gamma)} \, \mathrm{d}\boldsymbol{x} - \Psi_u^{2(2+\gamma)} v = \int_V |\Psi|^{2(2+\gamma)} \, \mathrm{d}\boldsymbol{x} - v \left[\frac{1}{v} \int_V (|\Psi|^2 - \Psi_\infty^2) \, \mathrm{d}\boldsymbol{x} + \Psi_\infty^2 \right]^{2+\gamma} \tag{27}$$

so that in nondimensional units

$$\mathcal{E} = \tfrac{1}{2} \int_V |\nabla\Psi|^2 \, \mathrm{d}\boldsymbol{x} + \tfrac{1}{4} \int_V (1 - |\Psi(\boldsymbol{x})|^2) V(|\boldsymbol{x} - \boldsymbol{x}'|)(1 - |\Psi(\boldsymbol{x}')|^2) \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{x}$$
$$+ \frac{\chi}{2(2+\gamma)} \left[\int_V |\Psi|^{2(2+\gamma)} \, \mathrm{d}\boldsymbol{x} - \left(\frac{\mathcal{M}}{v} + 1 \right)^{2+\gamma} v \right] \tag{28}$$

where the excess mass, $\mathcal{M}$, is given by

$$\mathcal{M} = \int_V (|\Psi|^2 - 1) \, \mathrm{d}\boldsymbol{x}. \tag{29}$$

For $\gamma = 1$ (27) can be further simplified by writing

$$\int_V |\Psi|^6 \, \mathrm{d}\boldsymbol{x} - \Psi_u^6 v = \int_V (|\Psi|^2 - \Psi_\infty^2)^3 \, \mathrm{d}\boldsymbol{x} - (\Psi_u^2 - \Psi_\infty^2)^3 v$$
$$+ 3\Psi_\infty^2 \left[\int_V (\Psi_\infty^2 - |\Psi|^2)^2 - (\Psi_\infty^2 - \Psi_u^2)^2 v \right]. \tag{30}$$

The terms $(\Psi_\infty^2 - \Psi_u^2)^2 v$ and $(\Psi_\infty^2 - \Psi_u^2)^3 v$ are $\mathrm{O}(1/v)$ and vanish as $v \to \infty$. In this limit, the dimensionless energy becomes

$$\mathcal{E}_1 = \tfrac{1}{2} \int_V |\nabla\Psi|^2 \, \mathrm{d}\boldsymbol{x} + \tfrac{1}{4} \int_V (1 - |\Psi(\boldsymbol{x})|^2) V(|\boldsymbol{x} - \boldsymbol{x}'|)(1 - |\Psi(\boldsymbol{x}')|^2) \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{x}$$
$$+ \frac{\chi}{6} \left[\int_V (|\Psi|^2 - 1)^2 (2 + |\Psi|^2) \, \mathrm{d}\boldsymbol{x} \right]. \tag{31}$$

Expressions (31) and (28) give the energy per unit length of the line vortex in dimensional units as

$$\mathcal{E}_1 = \frac{\kappa^2 \rho_\infty}{4\pi} \left(\int_0^\infty \left[\left(\frac{\mathrm{d}R}{\mathrm{d}r} \right)^2 + \frac{R^2}{r^2} \right] r \, \mathrm{d}r + \frac{\pi^{3/2}}{A} \int_0^\infty \int_0^\infty (1 - R^2(r)) \exp[-A^2(r^2 + r'^2)] \right.$$
$$\left. \times [g_0 I_0(\sigma) - g_1 I_1(\sigma)](1 - R^2(r')) r r' \, \mathrm{d}r' \, \mathrm{d}r + \frac{\chi}{3} \int_0^\infty (R^2 - 1)^2 (2 + R^2) r \, \mathrm{d}r \right) \tag{32}$$

and

$$\mathcal{E}_\gamma = \frac{\kappa^2\rho_\infty}{4\pi}\left(\int_0^\infty \left[\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2 + \frac{R^2}{r^2}\right] r\,\mathrm{d}r + \frac{\pi^{3/2}}{A}\int_0^\infty\int_0^\infty (1-R^2(r))\exp[-A^2(r^2+r'^2)]\right.$$
$$\times[g_0 I_0(\sigma) - g_1 I_1(\sigma)](1-R^2(r'))rr'\,\mathrm{d}r'\,\mathrm{d}r$$
$$\left. + \frac{\chi}{2+\gamma}\lim_{a\to\infty}\left[\int_0^a R^{2(2+\gamma)} r\,\mathrm{d}r - \left(\int_0^a R^2 r\,\mathrm{d}r\right)^{2+\gamma}\left(\frac{a^2}{2}\right)^{-(1+\gamma)}\right]\right). \tag{33}$$

The second term in the first integral in (32) and (33) represents the classical kinetic energy that diverges. This can be remedied by introducing a cut-off distance b, corresponding to the characteristic size of the container, so that the energy per unit length of the line vortex can be expressed in the form

$$\mathcal{E} = \frac{\kappa\rho_\infty}{4\pi}\left[\ln\left(\frac{b}{L}\right) - c\right] \tag{34}$$

where L is the healing length and for our model $L = 0.471\sqrt{1+\chi\gamma}$ Å, and c can be determined numerically from (32) or (33).

4. Large vortex rings

We consider solitary wave solutions of (17) that correspond to circular vortex rings that propagate along the z-axis with nondimensional velocity U preserving their forms. The wavefunction of such solitary waves satisfies the following equation:

$$2\mathrm{i}U\frac{\partial\Psi}{\partial z} = \nabla^2\Psi + \Psi\left(1 - \int |\Psi(\boldsymbol{x}')|^2 V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' - \chi|\Psi|^{2(1+\gamma)}\right). \tag{35}$$

We can perform a variation $\Psi \to \Psi + \delta\Psi$ in the expressions for momentum

$$\boldsymbol{p} = \frac{1}{2\mathrm{i}}\int[(\Psi^* - 1)\boldsymbol{\nabla}\Psi - (\Psi - 1)\boldsymbol{\nabla}\Psi^*]\,\mathrm{d}\boldsymbol{x} \tag{36}$$

and energy (28) and using (35) show that $\delta\mathcal{E} = U\delta p$, or

$$U = \frac{\partial\mathcal{E}}{\partial p}. \tag{37}$$

For a vortex ring of large radius R the results for the straight-line vortex can be used to give (see papers I and IV) the energy and momentum of such a ring as

$$\mathcal{E} = \frac{1}{2}\kappa^2\rho_\infty R\left[\ln\left(\frac{8R}{L}\right) - 2 + c\right] \tag{38}$$

and

$$p = \kappa\rho_\infty\pi R^2. \tag{39}$$

After differentiating $\mathcal{E}$ and p with respect to R and substituting into (37) we get the expression for the velocity of the large vortex ring as

$$U = \frac{\kappa}{4\pi R}\left[\ln\left(\frac{8R}{L}\right) - 1 + c\right]. \tag{40}$$

Glaberson and Donnelly (1986) used the experimental results of Rayfield and Reif (1964) on the relation between the energy and velocity of large vortex rings to estimate the vortex core parameter L. These estimates were based on the hollow core vortex model with $c = 0$ and produced $L \approx 0.81$ Å. Jones (1993) did similar calculations for the nonlocal model (1) with the potential (12a) and found $c = -0.13$, so that $L \approx 0.71$ Å for the optimal choice of the parameter A. For the local model (7) with $c = 0.381$ the vortex core parameter is $L \approx 1.19$ Å. These values of L are much larger that the healing length found from the sound speed, which is 0.47 Å for any of the above models. Jones (1993) posed the question of whether a self-consistent theory is possible, i.e., one where the vortex core parameter and the healing length are brought into harmony. The answer is 'Yes'. Our model (17) is able to bring about agreement. For $\gamma = 1$, $\chi = 3.5$, $A = 1.6$, $B = 1$, and $\eta = 1$ we numerically integrated (22) to find $c = 0.1825$, so that $L \approx 1$ Å, which is the healing length of our model. This gives the energy of a vortex ring travelling at 27 cm s^{-1} as 10 eV, which agrees with the experiments of Rayfield and Reif (1964).

5. Rarefaction waves

Jones and Roberts (paper IV) determined the entire sequence of solitary waves numerically for the local condensate model (7). They calculated the energy $\mathcal{E}$ and momentum p and showed how the location of the sequence in the $\mathcal{E}$–p plane relates to the superfluid helium dispersion curve. They found two branches meeting at a cusp where p and $\mathcal{E}$ assume their minimum values, p_m and $\mathcal{E}_m$. As $p \to \infty$ on each branch, $\mathcal{E} \to \infty$. On the lower branch the solutions were asymptotic to the large vortex rings of section 4. Since the local model has a healing length (based on the sound velocity) different from the vortex core parameter there are two possible ways to introduce dimensional units and to plot the solitary wave sequence next to the Landau dispersion curve on the $\mathcal{E}$–p plane. If the dimensional units based on the vortex core parameter are chosen, the cusp lies just above the Landau dispersion curve; if instead the healing length (sound speed) is selected the cusp meets the dispersion curve of the local model, which (we recall) does not have a roton branch.

As $\mathcal{E}$ and p decrease from infinity along the lower branch, the solutions begin to lose their similarity to large vortex rings, and (38)–(40) determine $\mathcal{E}$, p, and U less and less accurately, although (37) still holds. Eventually, for a momentum p_0 slightly greater than p_m, the rings lose their vorticity (Ψ loses its zero), and thereafter the solitary solutions may better be described as 'rarefaction waves'. The upper branch consists entirely of these and, as $p \to \infty$ on this branch, the solutions asymptotically approach the rational soliton solution of the Kadomtsev–Petviashvili (KP) equation.

In this section we investigate the rarefaction wave sequence of the nonlocal model (17). We only consider the limit of small amplitude solitary waves ($p \to \infty$). For simplicity we suppose that $\gamma = 1$. We substitute $\Psi = f + \mathrm{i}g$, where

$$\begin{aligned} f &= 1 + \epsilon^2 f_1 + \epsilon^4 f_2 + \cdots \\ g &= \epsilon g_1 + \epsilon^3 g_2 + \cdots \end{aligned} \tag{41}$$

into (35) and separate real and imaginary parts. We stretch the independent variables

$$\xi = \epsilon^2 x \qquad \eta = \epsilon^2 y \qquad \zeta = \epsilon z$$

and expand U in powers of ϵ as

$$U = U_0 + \epsilon^2 U_1 + \epsilon^4 U_2 + \cdots . \tag{42}$$

Notice that the asymptotics of the integral on the right-hand side of (35) with the potentials (12a) or (12b) can be found as

$$\begin{aligned} \frac{1}{\epsilon^5}\int \phi(\xi', \eta', \zeta') U\left(\sqrt{\frac{(\xi-\xi')^2}{\epsilon^4} + \frac{(\eta-\eta')^2}{\epsilon^4} + \frac{(\zeta-\zeta')^2}{\epsilon^2}}\right) \mathrm{d}\zeta'\,\mathrm{d}\eta'\,\mathrm{d}\xi' \\ = \mu_1\phi(\xi,\eta,\zeta) + \epsilon^2\mu_2\phi_{\zeta\zeta}(\xi,\eta,\zeta) \end{aligned} \tag{43}$$

where

$$\mu_1 = \left[\alpha + \frac{3}{2}\beta + \frac{15}{4}\delta\right]\frac{\pi^{3/2}}{A^3} \qquad \mu_2 = \left[\frac{4\alpha + 10\beta + 35\delta}{16}\right]\frac{\pi^{3/2}}{A^5}. \tag{44}$$

From the bulk normalization condition (19), $\mu_1 = 1 - \chi$. To leading order, the real and imaginary parts of (35) give

$$2U_0\frac{\partial g_1}{\partial\zeta} = (1+\chi)(2f_1 + g_1^2) \tag{45}$$

$$-2U_0\frac{\partial f_1}{\partial\zeta} = -\frac{\partial^2 g_1}{\partial\zeta^2} + (2f_1 + g_1^2)g_1(1+\chi) \tag{46}$$

so that

$$U_0 = \sqrt{(1+\chi)/2} \qquad 2f_1 + g_1^2 = \sqrt{2(1+\chi)}\partial g_1/\partial\zeta. \tag{47}$$

To the next order we have

$$\begin{aligned}\sqrt{2(1+\chi)}g_2' + 2U_1g_1' &= -f_1'' + (1+\chi)(2f_2 + 2g_1g_2 + f_1^2)\\ &\quad + (1+\chi)f_1(2f_1+g_1^2) + \mu_2(2f_1+g_1^2)'' + \chi(2f_1+g_1^2)^2\end{aligned} \tag{48}$$

and

$$\begin{aligned}-\sqrt{2(1+\chi)}f_2' - 2U_1f_1' &= -g_2'' - \nabla_H^2 g_1 + g_2(1+\chi)(2f_1+g_1^2)\\ &\quad + (1+\chi)g_1(2f_2 + 2g_1g_2 + f_1^2) + g_1\mu_2(2f_1+g_1^2)'' + \chi(2f_1+g_1^2)^2\end{aligned} \tag{49}$$

where primes denote the partial derivative in ζ and $\nabla_H^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$. If we define $D_1 = 2f_2 + 2g_1g_2 + f_1^2$ and $D_2 = \partial(g_2 - f_1 - 1/3g_1^3)/\partial\zeta$ then from (45)–(47)

$$\begin{aligned}U_0D_1 - D_2 &= \frac{U_1}{U_0}g_1' - \frac{1}{2}\chi g_1' g_1^2 + \frac{\chi^3 + 5\chi^2 - 2\chi - 2}{2\sqrt{2}(1+\chi)^{3/2}}(g_1')^2\\ &\quad + \frac{\chi}{\sqrt{2}\sqrt{1+\chi}}g_1g_1'' + \frac{1-2\mu_2}{2(1+\chi)}g_1'''\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial}{\partial\zeta}(U_0D_1 - D_2) &= \nabla_H^2 g_1 - \frac{\chi}{U_0}g_1g_1' - \frac{(\chi-1)\chi}{\chi+1}g_1(g_1')^2 - \frac{U_1}{U_0}g_1''\\ &\quad - \frac{\chi}{2}g_1^2g_1'' + U_0(4+\chi)g_1'g_1'' + \frac{\chi}{\sqrt{2}\sqrt{1+\chi}}g_1g_1^{(3)}.\end{aligned} \tag{51}$$

Consistency of these equations leads to a KP-type equation for the function g_1:

$$\begin{aligned}2\sqrt{2}U_1g_1'' - \sqrt{1+\chi}\nabla_H^2 g_1 + \frac{\partial}{\partial\zeta}\left[\frac{1-2\mu_2}{2\sqrt{1+\chi}}g_1^{(3)} - \frac{\sqrt{2}(3+5\chi)}{2(1+\chi)}(g_1')^2 + \frac{\sqrt{2}}{2}\chi g_1^2\right]\\ - \frac{2\chi}{\sqrt{1+\chi}}(g_1')^2 g_1 = 0.\end{aligned} \tag{52}$$

The corresponding equation governing g_1 for the local model (7) was obtained by Jones and Roberts (paper IV) and can be recovered from (52) by taking $\chi = 0$, $\mu_2 = 0$. The KP equation usually arises in studies of the propagation of sound waves in a weakly dispersive medium (for discussion and references see Kuznetsov and Musher 1986). Two types of the KP equations are distinguished. If the expansion of the dispersion law in the long-wave region is convex ($\omega''(k) > 0$ for small k) the dispersion is positive and the coefficients of the highest order term and the ∇_H^2 term have opposite signs; this corresponds to the KP I equation. If the dispersion relation is concave at small k the dispersion is called negative and the coefficients have the same sign; this leads to the KP II equation.

Superfluid ^{4}He has positive dispersion if the pressure lies below some threshold, and has negative dispersion above that threshold. This difference changes the wave dynamics dramatically. The most important change is that the KP II equation has no multidimensional solitons, while the KP I equation possesses such solutions, although they are unstable in three dimensions, as can be shown from Lyapunov's theorem (Kuznetsov and Musher 1986). The nonlocal model (17) with the potential (12*a*) has negative dispersion at low momenta and (52) is a KP II equation ($\mu_2 > \frac{1}{2}$), which does not have a solitary wave solution in the limit of $U \to U_0$. At low pressures the potential (12*b*) can be used and (52) becomes a KP I equation with an upper branch of solutions similar to the local model.

6. Stretched dipole moment, impulse and energy for solitary waves

In this section we define the stretched dipole moment and impulse for the solitary wave solution of (35) and obtain some integral properties of (35), that will be used as a check on numerical accuracy in section 7. To accomplish this we follow similar derivations of Jones and Roberts (paper IV) for the local model.

The flow at large distances from a classical vortex ring is dipolar; similarly, the solutions of (35) as $|\boldsymbol{x}| \to \infty$ has the form

$$\Psi \approx 1 - \frac{\mathrm{i}mz}{[z^2 + (1-2U^2)r^2]^{3/2}} + \cdots \tag{53}$$

where m is called 'the stretched dipole moment' of the wave. We relate m to the momentum, p, given in (36) by

$$4\pi m = p + U \int \left[1 - |\Psi|^2 - \mathrm{Re}\,(\Psi)\left(1 - \int |\Psi(\boldsymbol{x}')|^2 V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' - \chi|\Psi|^{2(1+\gamma)}\right)\right]\mathrm{d}\boldsymbol{x}. \tag{54}$$

Next we may replace Ψ by $\Psi - 1$ in the first integral of (28), integrate by parts discarding the vanishing surface integral, and obtain

$$\mathcal{E} = pU + \tfrac{1}{4}\int \left(1 - \int |\Psi(\boldsymbol{x}')|^2 V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' - \chi|\Psi|^{2(1+\gamma)}\right)|1-\Psi|^2\,\mathrm{d}\boldsymbol{x}. \tag{55}$$

7. Vortex rings of small radius

In this section we address the problem of numerically finding the sequence of vortex rings of small radius. This research has been motivated by the famous conjecture of Onsager (Donnelly 1974) that the roton can be pictured as 'the ghost of a vanished vortex ring'. So we would like to know the $\mathcal{E}$–p configuration of the sequence of such vortex rings and, starting from the moment the circulation disappears, the configuration of rarefaction pulses. We solved (17) numerically in the three-dimensional axisymmetric case in a frame of reference moving with the velocity U_F:

$$\begin{aligned}-2\mathrm{i}\Psi_t + 2\mathrm{i}U_F\Psi_z = \Psi_{rr} + \frac{1}{r}\Psi_r + \Psi_{zz} + \Psi\Bigg(1 - \chi|\Psi|^{2(1+\gamma)} - \int_0^\infty\int_{-\infty}^\infty\int_0^{2\pi} |\Psi(r',z')|^2 \\ \times U\left(\sqrt{r^2 + r'^2 - 2rr'\cos(\theta-\theta') + (z-z')^2}\right) r'\,\mathrm{d}\theta'\,\mathrm{d}z'\,\mathrm{d}r'\Bigg).\end{aligned} \tag{56}$$

The velocity U_F is chosen to retain the minimum of $|\Psi|$ at the centre of the frame. We used finite differences and the Raymond–Kuo radiation boundary condition (Raymond and Kuo 1986) allowing the outgoing radiation to escape from the integration box. Details of the numerics are discussed by Berloff (1999). We started our calculations from the large vortex ring ($R = 20$) with the core of the rectilinear vortex found in section 3, so that the initial configuration is close to the exact solution.

Nore *et al* (1993) studied the acoustic behaviour of the local model and demonstrated that the dispersive effects due to the quantum stress tensor become noticeable for some range of the width-to-height ratio of the travelling pulse. Similar dispersion takes place in the nonlocal model (56). To minimize this, we introduce a small dissipation into equation (56). The most physically relevant way of doing this for the local model was suggested by Carlson (1996). In real helium, even in the low-temperature range, normal fluid is present that is coupled to the superfluid and, through its viscosity, provides a high wavenumber sink. When modified to include mutual friction with the normal fluid, the superfluid Euler equation becomes (e.g., Khalatnikov 1965)

$$\dot{\boldsymbol{v}}_s + \boldsymbol{\nabla}\frac{v_s^2}{2} = -\boldsymbol{\nabla}(\mu - \zeta_3 \boldsymbol{\nabla}\cdot(\boldsymbol{j} - \rho \boldsymbol{v}_n) - \zeta_4 \boldsymbol{\nabla}\cdot \boldsymbol{v}_n) \tag{57}$$

where $\boldsymbol{j} = \rho \boldsymbol{v}$ is the mass current, μ is the chemical potential, $\boldsymbol{v}_n$ is the velocity of the normal fluid, and ζ_3 and ζ_4 are the coefficients of bulk viscosity. When the first dissipative term of (57) is introduced into the model (56) via the Madelung transform the nonlocal model with dissipation becomes

$$\begin{aligned} -2\mathrm{i}\Psi_t + 2\mathrm{i}U_F\Psi_z = \Psi_{rr} + \frac{1}{r}\Psi_r + \Psi_{zz} + \Psi\Bigg(1 - \chi|\Psi|^{2(1+\gamma)} - \int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} |\Psi(r', z')|^2 \\ \times U\left(\sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta') + (z - z')^2}\right) r'\,\mathrm{d}\theta'\,\mathrm{d}z'\,\mathrm{d}r'\Bigg) \\ -2\zeta_3\Psi\left(\frac{\partial}{\partial t} + \boldsymbol{v}_n\cdot\nabla\right)|\Psi|^2. \end{aligned} \tag{58}$$

We assume that the normal fluid is at rest in the reference frame moving with the ring. In our calculations ζ_3 was taken to be of order 10^{-2}. First, the initial field was evolved according to the dissipative model (58) until the dispersive effects were sufficiently minimized. After that, ζ_3 was set as zero and the calculations were continued using the nondissipative model (56). We emphasize that the solutions we present below satisfy the non-dissipative Hamiltonian system. After the emission of acoustic waves the system settles down to a solitary solution travelling with constant speed. The fact that the system reached the solution of (35) is also confirmed by the equality of two expressions (28) and (55) for the energy.

After the solitary wave solution is reached and the values of the stretched momentum, impulse, radius, and energy are recorded, we use this solution as the initial condition for the equation with dissipation (58). Dissipation slows down the motion of the vortex ring and reduces its radius. After that, we again set the dissipative parameter to zero and continue calculations for that new starting radius, thereby obtaining another solution on the vortex ring sequence.

In table 1 we show $\mathcal{E}$ as calculated from (32), p from (36), and m from (54), for equation (17) with the potential (12*b*). The values of parameters in (12*b*) were taken as $\chi = 0.2$, $A = 0.7$, $B = 0.5$, $\alpha \approx 11.8539$, $\beta \approx -10.5595$, $\delta \approx 4.7217$, and $\eta \approx -4.9824$, so that in (34) $c \approx 0.009$ close to the hollow core vortex. The value of p was checked by computing (55). Figure 2 gives the plot of the ring with radius $R = 12.6$ in terms of the density ρ. As the speed of the vortex ring approaches the Landau critical velocity, defined as $U_L = \omega_L/k_L$ where (k_L, ω_L) are the points on the dispersion curve where $\omega/k = \mathrm{d}\omega/\mathrm{d}k$, and which in our units is $U_L \approx 0.19$ (which corresponds to 58.4 m s^{-1}), it apparently loses its stability and evanesces into sound waves. In figure 3 we show the time evolution of the

Table 1.

U	$\mathcal{E}$	p	m	R
0.14	690	3390	270	13.1
0.15	646	3056	248	12.58
0.16	589	2685	218	11.89
0.167	521	2260	181	10.74
0.174	474	1990	159	9.93
0.189	417	1657	132	8.64

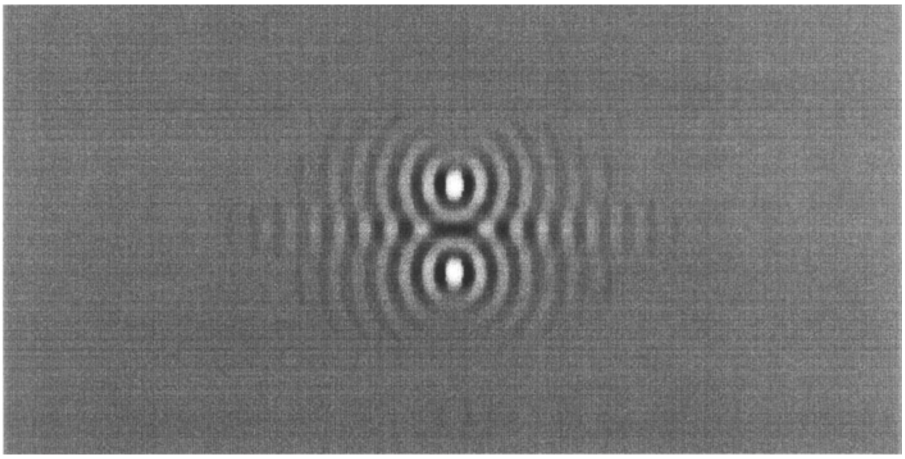

Figure 2. The density plot of the cross section of the axisymmetric vortex ring with radius $R = 12.6$. The darker regions correspond to an increase in density.

momentum and energy of an axisymmetric solution of equation (17) with the potential (12*b*) having $p = 1424$, $\mathcal{E} = 373$, so that the starting velocity of the ring is slightly above the Landau critical velocity. The total momentum and energy in the system are conserved, so the observed decay in p and $\mathcal{E}$ is balanced by energy radiated out of the computational boxes by sound waves. In figure 4, we show the time evolution of the same initial state by means of contour plots of the scaled density $|\Psi|^2$.

To understand this instability better, let us first imagine that normal fluid is present, and is at rest in the computational frame in which the vortex is stationary. Therefore, at infinity there is a uniform counterflow $\boldsymbol{U}$ $(= \boldsymbol{v}_n - \boldsymbol{v}_s)$ between normal and superfluid components, where $\boldsymbol{U} = U\boldsymbol{l}_z$ is the velocity of the vortex in the laboratory frame. In the absence of friction the dispersion relationship for the superfluid at infinity is merely Doppler shifted, i.e., the frequency (in the computational frame) for the wave with wavevector $\boldsymbol{k}$ is $\omega_D = \omega - \boldsymbol{U} \cdot \boldsymbol{k}$, where $\omega(k)$ is given by the dispersion relation for stagnant superfluid. If $U > U_L$, then $\omega_D < 0$ for all $\boldsymbol{k}$ in some neighbourhood of $\boldsymbol{k}_L = k_L\boldsymbol{l}_z$. Let us now recognize the existence of the friction between components that always exists at finite temperature. Let us model that friction using (58) and suppose that $\zeta_3 \ll \omega_L/k_L^3$. It is easily shown from (58) that, if $\Psi = 1 + \epsilon\psi$ where $\epsilon \ll 1$ and $\mathrm{Re}\,(\psi) = \exp[\mathrm{i}(\bar{\omega}t - \boldsymbol{k}\cdot\boldsymbol{x})]$, then

$$\bar{\omega} \approx \omega_D + \frac{\mathrm{i}\zeta_3 k^2}{2\omega(k)}\omega_D. \tag{59}$$

It is clear that, if $U > U_L$, then $Im(\bar{\omega}) < 0$ in the neighbourhood of $\boldsymbol{k} = \boldsymbol{k}_L$ in which $\omega_D < 0$, so that those modes are unstable. It should also be remarked that, even when ζ_3 is

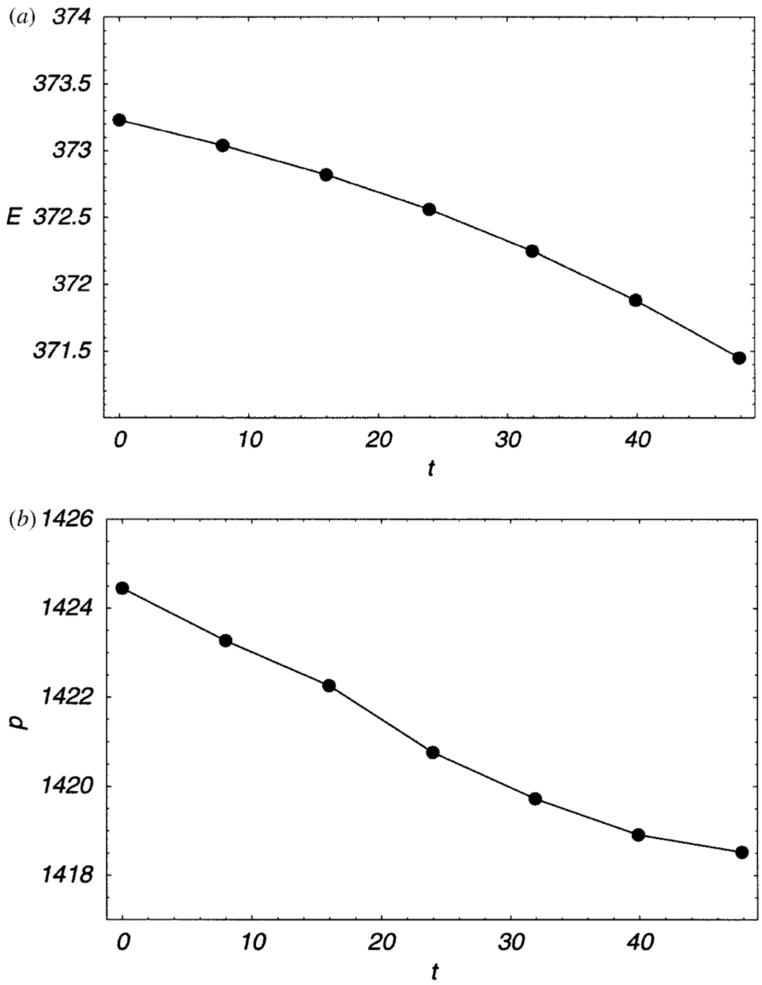

Figure 3. Time evolution of energy (*a*) and momentum (*b*) of an axisymmetric solution of equation (17) with the potential (12*b*) having $p = 1424$, $\mathcal{E} = 373$; the starting velocity is slightly above the Landau critical velocity.

zero, numerical dissipation will play its role in creating instability when $U > U_L$. So our vortex sequence necessarily terminates at the point where the speed of the vortex becomes the Landau critical velocity.

8. Conclusion

We derived a new model of superfluid helium ^{4}He with a realistic phonon–roton-like spectrum. Our strategy was to introduce as few phenomenological parameters as possible, so that the modifications made to the Ginsburg–Pitaevskii local model (7) are minimal. First, as was shown by Berloff (1999), if the δ-function potential is simply replaced by a nonlocal potential the resulting model possesses nonphysical mass concentrations, so that higher-order nonlinear terms have to be introduced to prevent the creation of such concentrations and the formation of catastrophic singularities. Such a higher-order nonlinearity was added following the ideas

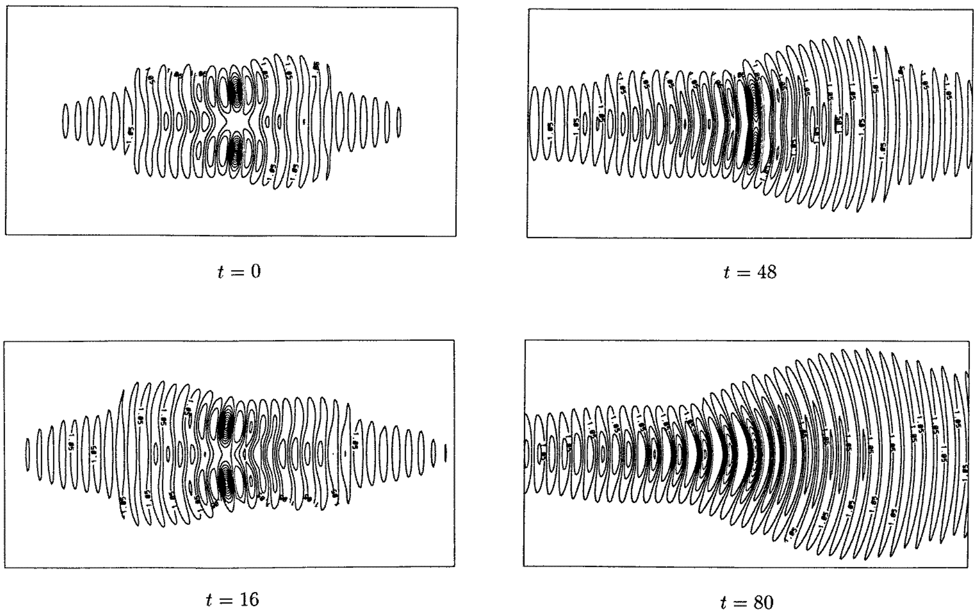

Figure 4. Time evolution of the initial state with $p = 1424$, $\mathcal{E} = 373$, with the starting velocity slightly above the Landau critical velocity as equidensity surfaces of the scaled density $|\Psi|^2$.

of density-functional theory. For this model we showed that the vortex core parameter and the healing length can be brought into agreement, so that the energy of the large vortex rings coincides with experimental observations.

Axisymmetric solutions for the resulting nonlocal Schrödinger-type equation (17) were analysed analytically and numerically. A family of axisymmetric solitary vortex rings has been derived numerically. An interesting result emerged from these numerical calculations: when the velocity of the vortex ring reaches the Landau critical velocity the ring becomes unstable and evanesces into sound waves. For any ring travelling with speed greater than the Landau critical velocity, the amplitude of the far-field solution will not decay exponentially at infinity, which makes the existence of such a ring impossible. One question that remains open is whether there is a cusp in the energy–momentum plane as occurs in the Ginzburg–Pitaevskii local model (7). The existence of an upper branch of rarefaction pulses having velocities that approach the velocity of sound was established for the potential that gives positive dispersion for small k. As these axisymmetric solutions are unstable, it is most probable that the entire upper branch of these solutions cannot be calculated by a time-stepping scheme. The alternative would be to solve the discretized equation (35) for specified values of U by Newton–Raphson iteration, and that requires matrix inversion at each iterative step. Because of the nonlocality of the potential in (35) the resulting matrices are not sparse, and this will lead to severe numerical difficulties. Also, the fate of the vortex ring as we proceed by increasing the velocity past the Landau critical velocity is not completely clear at the moment.

One of the goals of this paper is the same as that of an earlier paper in this sequence (paper IV): to clarify Onsager's concept of the roton as 'the ghost of a vanished vortex ring'. We hoped that the transition from the vortex ring to the sound pulse and the concomitant loss of vorticity would occur close to the roton minimum in energy–momentum space, or, possibly, close to the point where the group velocity and the phase velocity are equal (the Landau critical velocity U_L). Our calculations show that, indeed, there is a point on the p–$\mathcal{E}$ plane where the ring ceases to exist and where $U_L = \partial\mathcal{E}/\partial p$, but this point lies far from the roton minimum.

Finally, we make a speculative remark on how the idea of the roton as a ghostly vortex ring might be vindicated. As we have a great variety of potentials that lead to the Landau dispersion curve we can tune the parameters so that the line $\mathcal{E} = U_L p$, meets the p–$\mathcal{E}$ curve for the family of the vortex rings, to allow this sequence of vortex rings to be terminated at a lower energy and momentum level. Whether this process will lead to coalescence with the roton minimum is the subject of our future investigations.

Acknowledgments

NGB thanks the Trustees of the University of California for a President's Postdoctoral Fellowship during the tenure of which this research was initiated. We both thank the NSF for support from grant DMS-9803480 which was used to complete this work. We are grateful to Professor Chris Jones for helpful remarks and for providing us with a copy of his paper (Jones 1993) prior to its publication.

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